Certain right-angled Artin groups in mapping class groups

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Plan

(1) Introduction and statements of results

(2) Ideas of the proofs

The existence of embeddings between RAAGs
$\rightarrow$ Embeddings of RAAGs into MCGs (Main Theorem)
$\rightarrow$ Embeddings between MCGs (applications)

Right-angled Artin groups

Γ: a finite (simplicial) graph
$V(\Gamma) = \{v_1, v_2, \cdots, v_n\}$: the vertex set of Γ
$E(\Gamma)$: the edge set of Γ

Definition

The right-angled Artin group (RAAG) $A(\Gamma)$ on Γ is the group given by the following presentation:

$$A(\Gamma) = \langle v_1, v_2, \dots, v_n \mid [v_i, v_j] = 1 \text{ if } \{v_i, v_j\} \in E(\Gamma) \rangle.$$

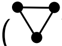
$A(\Gamma_1) \cong A(\Gamma_2)$ if and only if $\Gamma_1 \cong \Gamma_2$.

e.g.

$A($  $) \cong F_3$

$A($  $) \cong \mathbb{Z} * \mathbb{Z}^2$

$A($  $) \cong \mathbb{Z} \times F_2$

$A($  $) \cong \mathbb{Z}^3$

The mapping class groups of surfaces

$\Sigma := \Sigma^b_{g,p}$: the orientable surface of genus g with p punctures and b boundary components

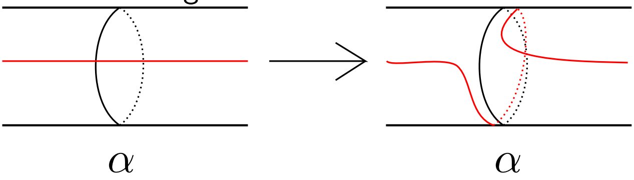
The mapping class group of Σ is defined as follows.

$$\mathrm{Mod}(\Sigma) := \mathrm{Homeo}_+(\Sigma, \partial\Sigma)/\text{isotopy}$$

$B_n := \mathrm{Mod}(\Sigma^1_{0,p})$ "the braid group on n strands"

α: an essential simple loop on $\Sigma^b_{g,p}$

The Dehn twist along α:



The curve graphs of surfaces

$\Sigma_{g,p}$: the orientable surface of genus g with p punctures
The curve graph $\mathcal{C}(\Sigma_{g,p})$ is a graph such that

- $V(\mathcal{C}(\Sigma_{g,p})) = \{$isotopy classes of escc on $\Sigma_{g,p}\}$
- escc α, β span an edge iff α, β can be realized by disjoint curves in $S_{g,p}$.

Fact (Subgroup generated by two Dehn twists)

Let α and β be non-isotopic escc on $\Sigma_{g,p}$.

(1) If $i(\alpha, \beta) = 0$, then the Dehn twists T_α and T_β generate $\mathbb{Z}^2 \cong A(\bullet\!\!-\!\!\!-\!\!\bullet)$ in $\mathrm{Mod}(\Sigma_{g,p})$.

(2) If $i(\alpha, \beta) = 1$, then T_α and T_β generate $\mathrm{SL}(2, \mathbb{Z})$ (when $(g, p) = (1, 0)$ or $(1, 1)$) or B_3 (otherwise).

(3) If the minimal intersection number of α and β is ≥ 2, then T_α and T_β generate $F_2 \cong A(\bullet \quad \bullet)$ (Ishida, 1996).

Theorem (Crisp–Paris, 2001)

If $i(\alpha, \beta) = 1$ and $\langle T_\alpha, T_\beta \rangle \cong B_3$, then T_α^2 and T_β^2 generate $F_2 \cong A(\bullet \quad \bullet)$ in $\mathrm{Mod}(\Sigma_{g,p})$.

Theorem (Koberda, 2012)

Γ: a finite graph, $\chi(\Sigma_{g,p}) < 0$.
If $\Gamma \leq \mathcal{C}(\Sigma_{g,p})$, then sufficiently high powers of "the Dehn twists $V(\Gamma)$" generate $A(\Gamma)$ in $\mathrm{Mod}(\Sigma_{g,p})$.

Here, a subgraph Λ of a graph Γ is said to be full if $\{u, v\} \in E(\Lambda) \Leftrightarrow \{u, v\} \in E(\Gamma)$ for all $u, v \in V(\Lambda)$.
We denote by $\Lambda \leq \Gamma$ if Λ is a full subgraph of Γ.

Motivation

Note: for any finite graph Γ, there is a surface Σ such that $A(\Gamma) \hookrightarrow \mathrm{Mod}(\Sigma)$ by Koberda's theorem.

Problem (Kim–Koberda, 2014)

Decide whether $A(\Gamma)$ is embedded into $\mathrm{Mod}(\Sigma_{g,p})$.

Theorem (Birman–Lubotzky–McCarthy, 1983)

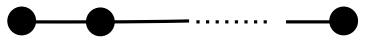
$A(K_n) \cong \mathbb{Z}^n \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if $n \leq 3g - 3 + p$.

Theorem (McCarthy, 1985)

$A(K_1 \sqcup K_1) \cong F_2 \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if $(g, p) \neq (0, \leq 3)$.

Theorem (Koberda, Bering IV–Conant–Gaster, K, 2017)

$F_2 \times F_2 \times \cdots \times F_2 \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ if and only if the number of the direct factors F_2 is at most $g + \lfloor \frac{g+p}{2} \rfloor - 1$.



P_n

P_n: the path graph on n vertices

The complement graph Γ^c of a graph Γ is the graph such that $V(\Gamma^c) = V(\Gamma)$ and $E(\Gamma^c) = \{\{u, v\} \mid \{u, v\} \notin E(\Gamma)\}$.

Main Theorem (K.–Kuno)

$A(P_m^c) \leq \mathrm{Mod}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

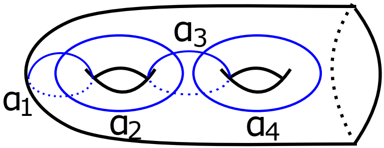
$$m \leq \begin{cases} 0 & ((g,p) = (0,0), (0,1), (0,2), (0,3)) \\ 2 & ((g,p) = (0,4), (1,0), (1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Some Applications

The homomorphisms $B_{2g+1} \to \mathrm{Mod}(\Sigma_{g,0}^1)$ and $B_{2g+2} \to \mathrm{Mod}(\Sigma_{g,0}^2)$, which map the generators of Artin type to the Dehn twists along a chain of interlocking simple closed curves, are injective by a theorem due to Birman–Hilden.

Case $B_{2g+1} = \langle \sigma_1, \ldots, \sigma_{2g} | \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \ [\sigma_i, \sigma_j] = 1 \rangle$;

$$B_{2g+1} \to \mathrm{Mod}(\Sigma_{g,0}^1)$$
$$\sigma_i \to T_{\alpha_i}$$



Fact

- $B_{2g+1} \hookrightarrow \mathrm{Mod}(\Sigma^1_{g,0})$.
- $B_{2g+2} \hookrightarrow \mathrm{Mod}(\Sigma^2_{g,0})$.

Theorem (Castel, 2016)

Suppose that $g \geq 0$.

- $B_{2g+1} \hookrightarrow \mathrm{Mod}(\Sigma^1_{g',0})$ implies $g \leq g'$.
- $B_{2g+2} \hookrightarrow \mathrm{Mod}(\Sigma^2_{g',0})$ implies $g \leq g'$.

We obtain the following result as a corollary of Main Theorem.

Corollary A (K.–Kuno)

Suppose that $g \geq 0$. Then the following hold.

(1) If B_{2g+1} is virtually embedded into $\mathrm{Mod}(\Sigma^1_{g',0})$, then $g \leq g'$.

(2) If B_{2g+2} is virtually embedded into $\mathrm{Mod}(\Sigma^2_{g',0})$, then $g \leq g'$.

In the above corollary, we say that a group G is virtually embedded into a group H if there is a finite index subgroup N of G such that $N \leq H$.
Each of (1) and (2) extends corresponding Castel's result and is optimum.
Note: residual finiteness of the mapping class groups guarantees that a large supply of finite index subgroups of the mapping class groups.

We also obtain the following result as a corollary of Main Theorem.

Corollary B

Let g and g' be integers ≥ 2. Suppose that $\mathrm{Mod}(\Sigma_{g,p})$ is virtually embedded into $\mathrm{Mod}(\Sigma_{g',p'})$. Then the following inequalities hold:

(1) $3g + p \leq 3g' + p'$,

(2) $2g + p \leq 2g' + p'$.

It is easy to observe that, if $(3g + p, 2g + p) = (3g' + p', 2g' + p')$, then $(g, p) = (g', p')$. Namely, we have;

Corollary B'

Let g and g' be integers ≥ 2.
If $\exists H \leq \mathrm{Mod}(\Sigma_{g,p})$, $\exists H' \leq \mathrm{Mod}(\Sigma_{g',p'})$: finite index subgroups s.t. $H \hookrightarrow H'$ and $H \hookleftarrow H'$, then $(g, p) = (g', p')$.

Idea of Proof

Proof of corollary A (1/2)

Main Theorem (rephrased)

$A(P_m^c) \leq \mathrm{Mod}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

$$m \leq \begin{cases} 0 & ((g,p) = (0,0),(0,1),(0,2),(0,3)) \\ 2 & ((g,p) = (0,4),(1,0),(1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Proof of corollary A (2/2)

Corollary A (rephrased)

(1) If B_{2g+1} is virtually embedded into $\mathrm{Mod}(\Sigma^1_{g',0})$, then $g \leq g'$.

((2) can be treated similarly and so skipped.)

Proof.

Every finite index subgroup of B_{2g+1} contains a right-angled Artin group A, but $\mathrm{Mod}(\Sigma^1_{g',0})$ does not contain A if $g' \leq g - 1$.

- $A(P^c_{2g+1}) \hookrightarrow B_{2g+1}$ (Main Thm).
- If G contains a right-angled Artin group A, then any finite index subgroup N of G contains A.
- If $g' \leq g - 1$, then $A(P^c_{2g+1})$ is not embedded in $\mathrm{Mod}(\Sigma^1_{g',0})$ (Main Thm).

□

Proof of Main Theorem (1/6)

Main Theorem (rephrased)

$A(P_m^c) \leq \mathrm{Mod}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

$$m \leq \begin{cases} 0 & ((g,p) = (0,0), (0,1), (0,2), (0,3)) \\ 2 & ((g,p) = (0,4), (1,0), (1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Proof of Main Theorem (2/6)

Lemma (K.)

Suppose that $\chi(\Sigma_{g,p}) < 0$. Then $A(P_m^c) \hookrightarrow \mathrm{Mod}(\Sigma_{g,p})$ only if $P_m^c \leq \mathcal{C}(\Sigma_{g,p})$.

By this lemma, the problem

Problem

Decide whether $A(P_m^c)$ is embedded into $\mathrm{Mod}(\Sigma_{g,p})$.

is reduced into the following problem when $\chi < 0$:

Problem

Decide whether $P_m^c \leq \mathcal{C}(\Sigma_{g,p})$.

Proof of Main Theorem (3/6)

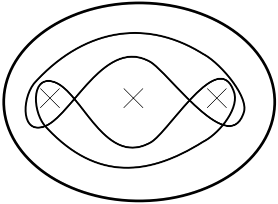
Problem (rephrased)

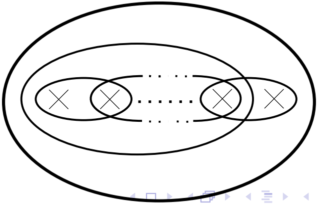
Decide whether $P_m^c \leq \mathcal{C}(\Sigma_{g,p})$.

A sequence $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ of closed curves on $\Sigma_{g,p}$ is called a linear chain if this sequence satisfies the following.

- Any two distinct curves α_i and α_j are non-isotopic.
- Any two consecutive curves α_i and α_{i+1} intersect non-trivially and minimally.
- Any two non-consecutive curves are disjoint.

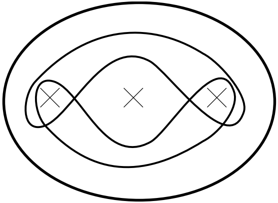
If $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a linear chain, we call m its length.



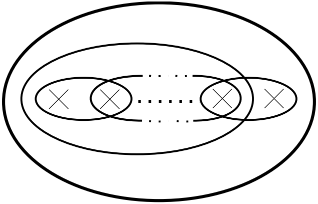


Proof of main Theorem (4/6)

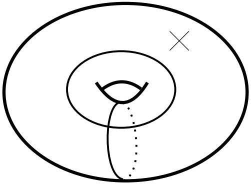
Note that if $|\chi(\Sigma_{g,p})| < 0$ and $\Sigma_{g,p}$ is not homeomorphic to neither $\Sigma_{0,4}$ nor $\Sigma_{1,1}$, then there is a linear chain of length m on $\Sigma_{g,p}$ if and only if $P_m^c \leq \mathcal{C}(\Sigma_{g,p})$.



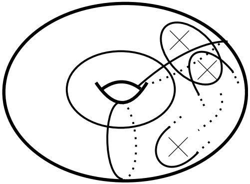
length 2



length $p - 1$

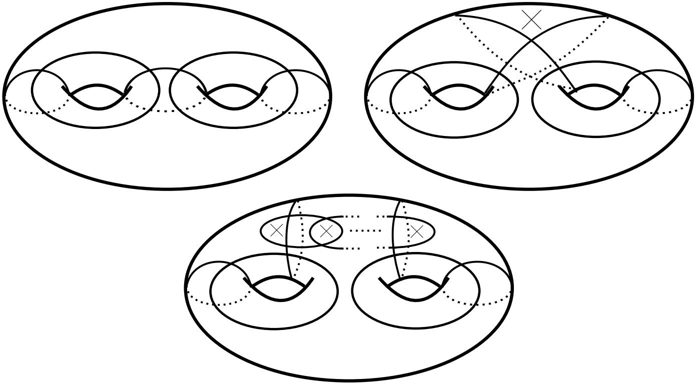


length 2



length $p + 2$

Proof of main Theorem (5/6)



length $2g + p + 1$

Proof of Main Theorem (6/6)

Main Theorem*

$P_m^c \leq \mathcal{C}(\Sigma_{g,p})$ if and only if m satisfies the following inequality.

$$m \leq \begin{cases} 0 & ((g,p) = (0,0), (0,1), (0,2), (0,3)) \\ 2 & ((g,p) = (0,4), (1,0), (1,1)) \\ p-1 & (g = 0,\ p \geq 5) \\ p+2 & (g = 1,\ p \geq 2) \\ 2g+p+1 & (g \geq 2) \end{cases}$$

Proof.

Double induction on the ordered pair (g, p). □

Distinguishing MCGs of the top comp = 3 surfaces

$3g - 3 + p = 3$ surfaces are $\Sigma_{0,6}$, $\Sigma_{1,3}$ and $\Sigma_{2,0}$.
It is well-known that the following sequence is exact:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Mod}(\Sigma_{2,0}) \to \mathrm{Mod}(\Sigma_{0,6}) \to 1.$$

This implies that $\mathrm{Mod}(\Sigma_{2,0})$ and $\mathrm{Mod}(\Sigma_{0,6})$ share many finite index subgroups.

Theorem (K.)

Suppose that (g, p) is either $(2, 0)$ or $(0, 6)$. Then any finite index subgroup of $\mathrm{Mod}(\Sigma_{g,p})$ is not embedded into $\mathrm{Mod}(\Sigma_{1,3})$.

$A(C_6^c) \hookrightarrow \mathrm{Mod}(\Sigma_{2,0})$ but $\mathrm{Mod}(\Sigma_{1,3})$ does not contain $A(C_6^c)$.

Thank you very much,
and we wish you a Merry Christmas!

T. Katayama and E. Kuno, “The RAAGs on the complement graphs of path graphs in mapping class groups”, preprint.
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