Complexes induced from spherical curves and distances derived from them

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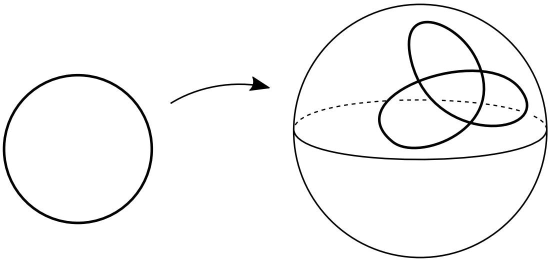
Joint work with M. Hashizume, N. Ito, T. Kobayashi, and H. Murai

1. Intro

Background

Spherical curve:

The image of a generic immersion of a circle into a 2-sphere.

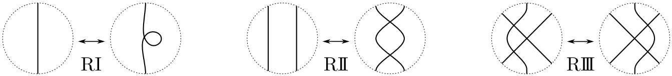


We identify each spherical curve with it's ambient isotopy class.

Background

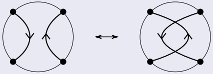
R-moves

We consider the following operations on the spherical curves.

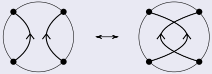


Further we devide RII, RIII as follows.

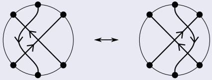
Definition 1



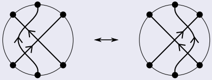
strong RII



weak RII



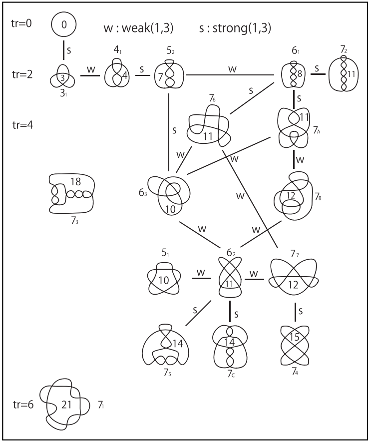
strong RIII



weak RIII

Background

N. Ito, Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on knot projections, *Osaka J. Math.* **52** (2015), 617–646.



$P \overset{s}{-} P'$ ($P \overset{w}{-} P'$) means

P' is obtained from P by **one** strong (weak) RIII and some RI's.

$\Downarrow$

このような 1-complex の構成方法は他にもある．

1-complex を定めるごとに，そこから誘導される spherical curve 間の distance が定まる．

$\Downarrow$

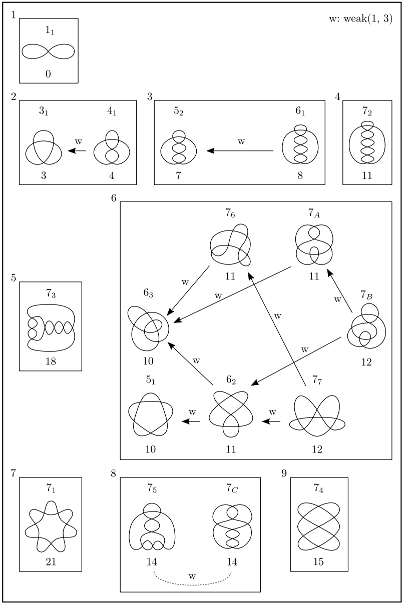
今回は，$P \overset{w}{-} P'$ に注目してこの distance について調べた．

Background

・N. Ito and Y. Takimura, (1, 2) and weak (1, 3) homotopies on knot projections, J. Knot Theory Ramifications 22 (2013), 1350085, 14pp.

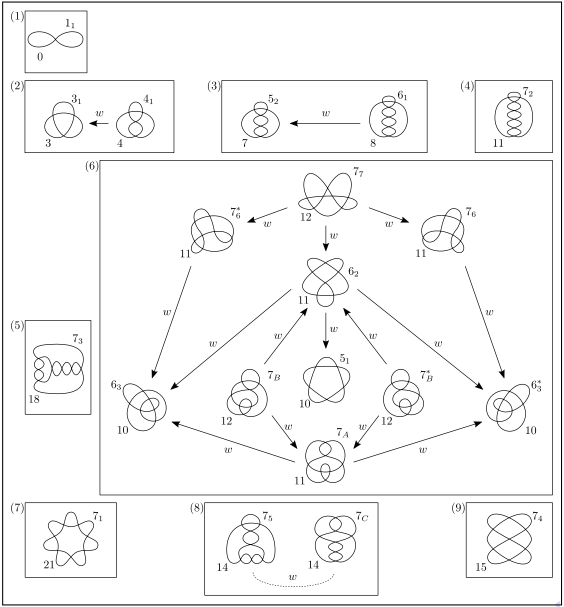
・N. Ito, Y. Takimura, and K. Taniyama, Strong and weak (1, 3) homotopies on spherical curves and related topics, Intelligence of Low-dimensional Topology 1960 (2015), 101–106.

7 交点以下の prime spherical curves は RI, weak RIII によって生成される equivalence relation により 9 個の equivalence class に分かれることがわかる.



$\mathcal{P}_{\leq 7}$ の weak (1,3) classes.

Background



The weak (1,3) equivalence classes of $\mathcal{P}_{\leq 7}$.

P: a spherical curve

Definition 2

P and P' are RI-equiv (denoted $P \sim_{RI} P'$) $\overset{\text{def}}{\Leftrightarrow}$
P' is obtained from P by a sequence of deformations of type RI.

A proposal of 1-complex derived from the spherical curves

$\mathcal{C}$: the set of the ambient isotopy classes of spherical curves

$$\begin{array}{ccc} \mathcal{C} & \to & \tilde{\mathcal{C}} = \mathcal{C}/\sim_{RI} \\ \Cup & & \Cup \\ P & \to & [P] \end{array}$$

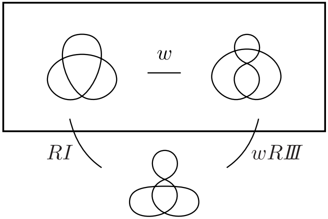
Let $\tilde{\mathcal{K}}_{w3}$ be the 1-complex s.t.

- the vertices of $\tilde{\mathcal{K}}_{w3}$ corresponds to $\tilde{\mathcal{C}}$
- two vertices $v, v' \in \tilde{\mathcal{K}}_{w3}$ are joined by an edge

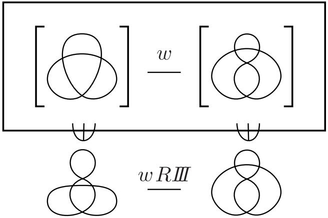
$\Leftrightarrow \exists P \in v$, $\exists P' \in v'$ s.t. P' is obtained from P by one weak RIII.

Background

I-T-T



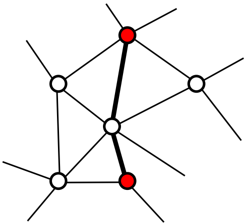
Our formulation



このとき，$\tilde{\mathcal{K}}_{w3}$ 上の path distance により，
$\tilde{\mathcal{C}}$ に distance $\tilde{d}_{w3}$ を導入する．

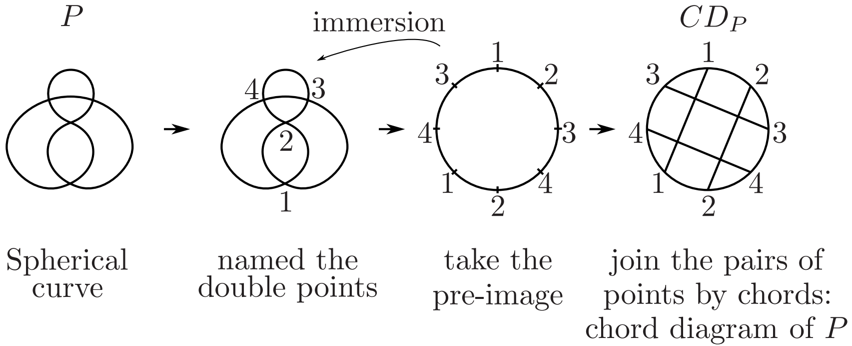
即ち，

$$\tilde{d}_{w3}(v,v') \\ = \begin{cases} \min\{\text{the number of the edges of } J \mid J\text{: path in } \tilde{\mathcal{K}}_{w3} \text{ joining } v \text{ and } v'\} \\ \infty \quad \text{if } \nexists \text{ path joining } v,\ v' \text{ in } \tilde{\mathcal{K}}_{w3}. \end{cases}$$



2. Preliminaries

Chord diagram derives from spherical curve



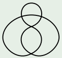
Chord diagram derives from spherical curve

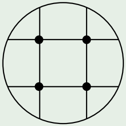
Then

$\bigotimes(P) \stackrel{\text{def}}{=}$ the number of subchords of CD_P s.t. 

$=$ the number of double points in (generic) CD_P.

Example 1

$\bigotimes \Big($  $\Big) = 4$

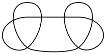


Further

$n(P) \overset{\text{def}}{=}$ the number of double points of P

(=the number of chords of CD_P)

$f_c(P) \overset{\text{def}}{=}$ the number of prime factors of P

$f_c\Big($  $\Big) = 2$

3. Main Results

Main Results

Proposition 1

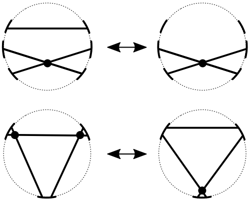
For spherical curves P, P' which are equivalent under RI and weak RIII, we have:

$$\bigotimes(P) - \bigotimes(P') \equiv \tilde{d}_{w3}([P],[P']) \ (mod\ 2)$$

$\because$

$$P_i \underset{\text{RI}}{\leftrightarrow} P_j \quad \Rightarrow \bigotimes(P_j) = \bigotimes(P_i)$$

$$P_i \underset{\text{wRIII}}{\longleftrightarrow} P_j \quad \Rightarrow \bigotimes(P_j) = \bigotimes(P_i) \pm 1$$



(cf. I-T-T)

Main Results

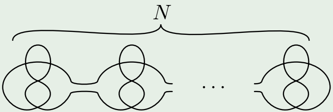
Theorem 1

Let P, P' be spherical curves.
Suppose $n(P) > n(P')$, $f_c(P) = f_c(P')$.
Then

$$\tilde{d}_{w3}([P],[P']) \geq n(P) - n(P').$$

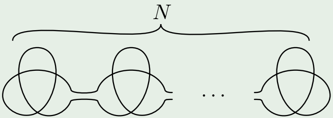
Example 1

P_N



$$n(P_N) = 4N$$

P'_N



$$n(P'_N) = 3N$$
$$f_c(P_N) = f_c(P'_N) = N$$

Main Results

Definition 3

P is RI-minimal $\overset{\text{def}}{\Leftrightarrow}$ P does not contain a 1-gon

Theorem 2

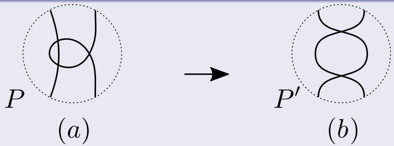
Let P, P' be RI-minimal spherical curves such that $f_c(P) - n(P) = f_c(P') - n(P')$. Suppose that $\tilde{d}_{w3}([P], [P']) = 1$.

Then P' is obtained from P by one weak RIII (no RI is necessary).

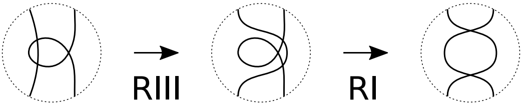
Main Results

Definition 4

Deformation of type α:



Fact. Definition of type α is realized by a RIII and RI.



We say that the deformation is type weak (strong resp.) α if the RIII is weak (strong resp.).

Main Results

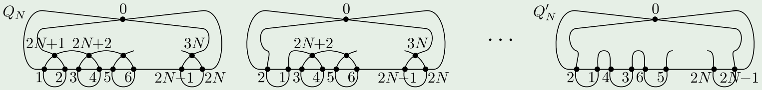
Theorem 3

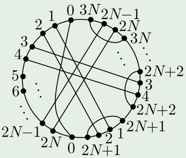
Let P, P' be RI-minimal spherical curves such that $f_c(P) = f_c(P') = 1$, and that $\tilde{d}_{w3}([P], [P']) = q < \infty$. Suppose that $n(P) - n(P') = q$, and that $\bigotimes(P) - \bigotimes(P') = q$.

Then P' is obtained from P by successively applying deformations of type weak α q times.
(This realizes $\tilde{d}_{w3}([P], [P'])$.)

Example 2

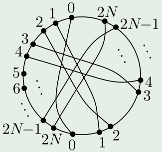
$N \in \mathbb{N}$, Q_N, Q'_N: spherical curves as in Figure.





$$
\begin{cases} f_c(Q_N) = 1 \\ n(Q_N) = 3N+1 \\ \bigotimes(Q_N) = 2N^2 + 2N \end{cases}
$$

$$
\begin{cases} f_c(Q'_N) = 1 \\ n(Q'_N) = 2N+1 \\ \bigotimes(Q'_N) = 2N^2 + N \end{cases}
$$

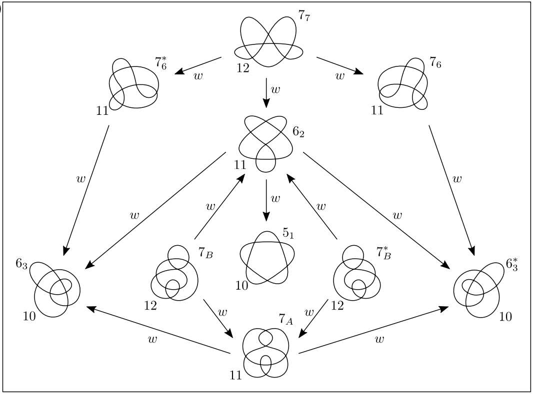


$\tilde{d}_{w3}([Q_N], [Q'_N]) = N.$

Application

Figure 3

(6)



is tautly embedded in $\tilde{\mathcal{K}}_{w3}$.

(i.e. For each pair P, P' in Figure 3, $\tilde{d}_{w3}([P],[P'])$ is realized by the minimal number of edges in paths in the complex of Figure 3 joining P and P'.)

Application

For Example:

We can show that $\tilde{d}_{w3}([7_6], [6_3]) = 3$ by applying Theorem 3.

In fact, since $f_c(7_6) = f_c(6_3) = 1$.
By Figure 3, we have $\tilde{d}_{w3}([7_6], [6_3]) \leq 3$.
If $\tilde{d} < 3$, then $\tilde{d} = 1$ by Proposition 2.
Assume $\tilde{d} = 1$.

Then by Theorem 2,
6_3 is obtained from 7_6 by weak RIII.
We can show that this is not the case,
by using the analysis as in Figure.